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A viscosity approximation method for equilibrium problems, fixed point problems of nonexpansive mappings and a general system of variational inequalities

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Abstract In this paper, we introduce and study a new iterative scheme for finding the common element of the set of common fixed points of a sequence of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of the general system of variational inequality for α and μ-inverse-strongly monotone mappings. We show that the sequence converges strongly to a common element of the above three sets under some parameters controlling conditions. This main theorem extends a recent result of Ceng et al. (Math Meth Oper Res 67:375–390, 2008) and many others.

Keywords Nonexpansive mapping · Equilibrium problem · Fixed point · General system of variational inequality

Mathematics Subject Classification (2000) 49J30 · 49J40 · 47H09 · 47H10

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively and let C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1}$$

The set of solutions of Eq. (1) is denoted by $EP(F)$. This equilibrium problem contains the fixed point problem, optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem as its special cases (see, e.g. , Blum and Oettli [2]). Numerous problems in physics, optimization, and economics reduce to find a solution of Eq. (1). Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then

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$z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0$$

for all $v \in C$. The variational inequality has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences; see, e.g. [5, 10, 18–21, 24, 26, 27] and the references therein. Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the current interest in functional analysis. It is natural to consider a unified approach to these different problem; see e.g. [10, 15, 19, 27]. A mapping A of C into H is called α-inverse-strongly monotone [3, 9] if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of S. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C. For finding an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [19] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{2}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$, α_n is a sequence in (0, 1), and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [10] and Zeng and Yao [27] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$. In 2006, Yao and Yao [25] introduced the following iterative scheme:

Let C be a closed convex subset of real Hilbert space H. Let A be an $\alpha-$ inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n Ax_n) \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n), \end{aligned} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in [0, 1] and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ defined by Eq. (3) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings under some parameters controlling conditions.

Motivated and inspired by Takahashi and Takahashi [18], Plubtieng and Punpaeng [12] introduce a new iterative process below for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for an α-inverse-strongly monotone mapping in a real Hilbert space. Suppose $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;\\ &y_n = P_C(u_n - \lambda_n A u_n)\\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n), \quad n \in \mathbf{N}. \end{aligned} \tag{4}$$

They proved that if the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ generated by Eq. (4) converges strongly to a common element of the set of fixed points of nonexpansive mappings and the solution of variational inequality and an equilibrium problems.

Let C be a closed convex subset of real Hilbert space H. Let $A, B : C \to H$ be two mappings. We consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C,\\ &\langle \mu B x^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{5}$$

which is called a general system of variational inequalities where $\lambda \geq 0$ and $\mu \geq 0$ are two constants. The set of solution of Eq. (5) is denoted by Ω. In particular, if $A = B$, then problem Eq. (5) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C,\\ &\langle \mu A x^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{6}$$

which is defined by Verma [20] (see also Verma [21]), and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem Eq. (6) reduces to the classical variational inequality $VI(A, C)$.

Very recently, Ceng et.al [4] introduce and study a relaxed extragradient method for finding a common of the set solution of Eq. (5) for the α and β-inverse strongly monotone and the set of fixed points of a nonexpansive mapping in real Hilbert space. Let $x_1 = u \in C$ and $\{x_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \mu B x_n)\\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), n \in \mathbf{N}. \end{aligned} \tag{7}$$

Then, they proved that the iterative sequence $\{x_n\}$ converges strongly to a solution of problem Eq. (5) Motivated by above results, Kumam and Kumam [8] introduce a viscosity relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solutions of a general system of variational inequality problem for inverse-strongly-monotone mappings. Then they proved strong convergence theorems under some parametric controlling conditions.

On the other hand, Aoyama, et al. [1] introduce a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \tag{8}$$

for all $n \in \mathbf{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in [0, 1] and $\{T_n\}$ is a sequence of nonexpansive mappings with some condition. They proved that $\{x_n\}$ defined by Eq. (8) converges strongly to a common fixed point of $\{T_n\}$.

Motivated and inspired by Plubtieng and Punpaeng [12] and Ceng et.al [4], this paper is organized as follows. In Sect. 2, we present some basic concepts and useful lemmas for proving the convergence result of this paper. In Sect. 3, we introduce the following iterative sequence in a Hilbert space H. Let C be a nonempty closed convex subset of H and let f be a contraction on H. Let $A, B : C \to H$ be the α and $\beta-$inverse-strongly monotone mappings, respectively. Given $x_0 = u \in C$ and

$$\begin{aligned} & F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ & y_n = P_C(u_n - \mu B u_n) \\ & x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(y_n - \lambda A y_n), \end{aligned} \tag{9}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{S_n\}$ is a sequence of nonexpansive mappings C into H with some conditions. Then we prove that the sequence $\{x_n\}$ defined by Eq. (9) converges strongly to a common of the set of common fixed points of a sequence of nonexpansive mappings, the set of solutions of an equilibrium problem, and the solution set of the general system variational inequality problem which is connected with the result of Ceng et.al [4]. In Sect. 4, we apply our main theorem to the W-mapping and a strictly pseudocontractive on C.

2 Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{10}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{11}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{12}$$

for all $x \in H$, $y \in C$.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 1 [11] *Let $(E, \langle ., .\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2 [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 3 [7] *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 4 ([23]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbf{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2]

Lemma 5 [2] *Let* C *be a nonempty closed convex subset of* H *and let* F *be a bifunction of* $C \times C$ *into* $\mathbf{R}$ *satisfying (A1)-(A4). Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \textit{for all} \quad y \in C.$$

The following lemma was also given in [6].

Lemma 6 [6] *Assume that* $F : C \times C \to \mathbf{R}$ *satisfies (A1)-(A4). For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $z \in H$. *Then, the following hold:*

1. T_r *is single-valued;*
2. T_r *is firmly nonexpansive,i.e., for any* $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;
3. $F(T_r) = EP(F)$;
4. $EP(F)$ *is closed and convex.*

Lemma 7 [4] *For given* $x^*, y^* \in C$, (x^*, y^*) *is a solution of problem* (5) *if and only if* x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x - \mu Bx)$.

Lemma 8 *In a real Hilbert space* H, *there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 9 [13] *Let* C *be a nonempty bounded closed convex subset of Hilbert space* H *and* $\{T_n\}$ *a sequence of mappings of* C *into itself. Suppose that*

$$\lim_{k,l\to\infty} \rho_l^k = 0, \tag{13}$$

where $\rho_l^k = \sup\{\|T_k z - T_l z\| : z \in C\} < \infty$ *, for all* $k, l \in \mathbf{N}$. *Then for each* $x \in C$, $\{T_n x\}$ *converges strongly to some point of* C. *Moreover, let* T *be a mapping from* C *in to itself defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in C.$$

Then $\limsup_{n\to\infty}\{\|Tz - T_n z\| : z \in C\} = 0$.

From lemma 9, it easy to see that T is nonexpansive.

3 Main results

In this section, we prove strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of a sequence of nonexpansive mappings and of the solution set of the general system variational inequality problem for an α and β-inverse-strongly monotone mapping in a real Hilbert space.

Theorem 1 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C \to \mathbf{R}$ *satisfying (A1)-(A4) and let* $A, B : C \to H$ *be the* α *and* $\beta-$*inverse-strongly monotone mappings, respectively. Let* f *be a contraction of* H *into itself with* $\alpha \in (0, 1)$ *and let* $\{S_n\}$ *be a sequence of nonexpansive mappings of* C *into* H *such that satisfies condition Eq.* (13) *and* $\cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap EP(F) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are given by*

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;\\ &y_n = P_C(u_n - \mu B u_n) \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(y_n - \lambda A y_n), \end{aligned} \tag{14}$$

for all $n \in \mathbf{N}$, *where* $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfying*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

If S *is a mapping of* C *into itself defined by* $Sx = \lim_{n\to\infty} S_n x$ $\forall x \in C$ *and* $F(S) = \cap_{n=1}^{\infty} F(S_n)$, *then* $\{x_n\}$ *converges strongly to* $\bar{x} \in F(S) \cap \Omega \cap EP(F)$ *where* $\bar{x} = P_{F(S)\cap\Omega\cap EP(F)} f(\bar{x})$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem Eq.* (5) *such that* $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Proof Put $Q = P_{F(S)\cap\Omega\cap EP(F)}$. It easy to see that Qf is a contraction. By Banach contraction principle, there exist $z_0 \in F(S)\cap\Omega\cap EP(F)$ such that $z_0 = Qf(z_0) = P_{F(S)\cap\Omega\cap EP(F)} f(z_0)$. Since $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$, it easy to see that $I - \lambda A$ and $I - \mu B$ are nonexpansive. Let $x^* \in F(S)\cap\Omega\cap EP(F)$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 6. Thus, we have $x^* = S_n x^* = P_C[P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)] = T_{r_n} x^*$. Put $y^* = P_C(x^* - \mu B x^*)$ and $v_n = P_C(y_n - \lambda A y_n)$. Then $x^* = P_C(y^* - \lambda A y^*)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n$ and hence

$$\begin{aligned} \|v_n - x^*\| &= \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\| \leq \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\| \\ &\leq \|y_n - y^*\| = \|P_C(u_n - \mu B u_n) - P_C(x^* - \mu B x^*)\| \\ &\leq \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\| \leq \|u_n - x^*\| \\ &= \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|. \end{aligned} \tag{15}$$

Thus, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - x^*\| \\ &\le \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|v_n - x^*\| \\ &\le \alpha_n \|f(x_n) - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n(1-\alpha))\|x_n - x^*\| \\ &\le \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\right\}.\end{aligned}$$

By induction, we get $\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{\|f(x^*)-x^*\|}{1-\alpha}\}$ for all $n \ge 1$. This implies that $\{x_n\}$ is bounded and hence the sets $\{u_n\}$, $\{v_n\}$, $\{S_{n+1}v_n\}$, $\{Bu_n\}$ and $\{Ay_n\}$ are also bounded. Moreover, we observe that

$$\begin{aligned}\|v_{n+1} - v_n\| &= \|P_C(y_{n+1} - \lambda A y_{n+1}) - P_C(y_n - \lambda A y_n)\| \\ &\le \|(y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n)\| \le \|y_{n+1} - y_n\| \\ &= \|P_C(u_{n+1} - \mu B u_{n+1}) - P_C(u_n - \mu B u_n)\| \\ &\le \|(u_{n+1} - \mu B u_{n+1}) - (u_n - \mu B u_n)\| \le \|u_{n+1} - u_n\|. \end{aligned} \tag{16}$$

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we note that

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \quad \text{for all} \quad y \in C \tag{17}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0 \quad \text{for all} \quad y \in C. \tag{18}$$

Putting $y = u_{n+1}$ in Eq. (17) and $y = u_n$ in Eq. (18), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0 \tag{19}$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0. \tag{20}$$

From (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \ge 0 \tag{21}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \ge 0. \tag{22}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbf{N}$. Thus, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\le \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\le \|u_{n+1} - u_n\| \left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\|\right\}\end{aligned}$$

and hence

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ \le \|x_{n+1}-x_n\| + \frac{1}{c}|r_{n+1}-r_n|L, \tag{23}$$

where $L=\sup\{\|u_n-x_n\|: n\in \mathbf{N}\}$. It follows from Eqs. (16) and (23) that

$$\|v_{n+1}-v_n\| \le \|x_{n+1}-x_n\| + \frac{1}{c}|r_{n+1}-r_n|L. \tag{24}$$

Let $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$. We note that

$$z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}=\frac{\alpha_n f(x_n)+\gamma_n S_n P_C(y_n-\lambda_n A y_n)}{1-\beta_n}=\frac{\alpha_n f(x_n)+\gamma_n S_n v_n}{1-\beta_n}$$

and hence

$$z_{n+1}-z_n=\frac{\alpha_{n+1} f(x_{n+1})+\gamma_{n+1} Sn+1 v_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n S_n v_n}{1-\beta_n} \\ =\frac{\alpha_{n+1} f(x_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)}{1-\beta_n}+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(S_{n+1}v_{n+1}-S_{n+1}v_n) \\ +\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)S_{n+1}v_n+\frac{\gamma_n}{1-\beta_n}(S_{n+1}v_n-S_n v_n). \tag{25}$$

Combining Eqs. (24) and (25), we have

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\ +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|S_{n+1}v_n\| \\ +\frac{\gamma_n}{1-\beta_n}\sup_{z\in\{v_n\}}\|S_{n+1}z-S_n z\|-\|x_{n+1}-x_n\| \\ \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\ +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{(1-\beta_{n+1})c}|r_{n+1}-r_n|L \\ +\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|S_{n+1}v_n\| \\ +\frac{\gamma_n}{1-\beta_n}\sup_{z\in\{v_n\}}\|S_{n+1}z-S_n z\|-\|x_{n+1}-x_n\| \\ \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\ +\frac{\gamma_{n+1}}{(1-\beta_{n+1})c}|r_{n+1}-r_n|L \\ +\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|S_{n+1}v_n\| \\ +\frac{\gamma_n}{1-\beta_n}\sup_{z\in\{v_n\}}\|S_{n+1}z-S_n z\|.$$

This together with (ii), (iii), (iv) and Eq. (13) imply that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 2, we obtain $\|z_n - x_n\| \to 0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0. \tag{26}$$

From (iv), Eqs. (16), (23) and (26), we also have $\|v_{n+1}-v_n\|\to 0$, $\|u_{n+1}-u_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$ as $n\to\infty$. Since

$$x_{n+1}-x_n = \alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n v_n - x_n = \alpha_n(f(x_n)-x_n)+\gamma_n(S_n v_n - x_n),$$

it follows by (ii) and Eq. (26) that $\|x_n - S_n v_n\|\to 0$ as $n\to\infty$. This implies that

$$\|Sv_n - x_n\| \le \|Sv_n - S_n v_n\| + \|S_n v_n - x_n\| \le \sup_{z\in\{v_n\}}\|Sz-S_nz\| + \|S_n v_n - x_n\| \to 0$$

For $x^*\in F(S)\cap\Omega\cap EP(F)$, we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle = \langle u_n - x^*, x_n-x^*\rangle\\ &= \frac{1}{2}(\|u_n-x^*\|^2+\|x_n-x^*\|^2-\|x_n-u_n\|^2)\end{aligned}$$

and hence $\|u_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-u_n\|^2$. From Eq. (16), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n v_n - x^*\|^2\\ &\le \alpha_n\|f(x_n)-f(x^*)\|^2+\alpha_n\|f(x^*)-x^*\|^2+\beta_n\|x_n-x^*\|^2\\ &\quad+\gamma_n\|S_n v_n - x^*\|^2\\ &\le \alpha_n\alpha\|x_n-x^*\|^2+\alpha_n\|f(x^*)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\ &\le(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\alpha_n\|f(x^*)-x^*\|^2+\gamma_n\|u_n-x^*\|^2\\ &\le \alpha_n\|f(x^*)-x^*\|^2+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2\\ &\quad+\gamma_n(\|x_n-x^*\|^2-\|x_n-u_n\|^2)\\ &= \alpha_n\|f(x^*)-x^*\|^2+(\alpha_n\alpha+\beta_n+\gamma_n)\|x_n-x^*\|^2-\gamma_n\|x_n-u_n\|^2\\ &\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|x_n-u_n\|^2\end{aligned} \tag{27}$$

and hence

$$\begin{aligned}\gamma_n\|x_n-u_n\|^2 &\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\end{aligned} \tag{28}$$

since $\alpha_n\to 0$ and Eq. (26) imply that $\|u_n-x_n\|\to 0$ as $n\to\infty$. This implies that

$$\|S_n v_n - u_n\| \le \|S_n v_n - x_n\| + \|x_n - u_n\| \to 0$$

and so

$$\|Sv_n - u_n\| \le \|Sv_n - S_n v_n\| + \|S_n v_n - u_n\| \le \sup_{z\in\{v_n\}}\|Sz-S_nz\| + \|S_n v_n - u_n\| \to 0$$

From Lemma 1, Eqs. (15) and (27), we get

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|^2\\
&\quad+\gamma_n\|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2\\
&\quad+\gamma_n\|y_n-\lambda Ay_n-y^*-\lambda Ay^*\|^2\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2\\
&\quad+\gamma_n\|(y_n-y^*)-\lambda(Ay_n-Ay^*)\|^2\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\big[\|y_n-y^*\|^2\\
&\quad-2\lambda\langle y_n-y^*,Ay_n-Ay^*\rangle+\lambda^2\|Ay_n-Ay^*\|^2\big]\\
&\le \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\big[\|y_n-y^*\|^2\\
&\quad+\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\big]\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\|y_n-y^*\|^2\\
&\quad+\gamma_n\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\
&\quad+\gamma_n\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|+\|x_n-x^*\|^2+\gamma_n\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-y^*\|^2\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2\\
&\quad+\gamma_n\|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2\\
&\quad+\gamma_n\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2\\
&\quad+\gamma_n[\|(u_n-x^*)-\mu(Bu_n-Bx^*)\|^2\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\big[\|u_n-x^*\|^2\\
&\quad-2\mu\langle u_n-x^*,Bu_n-Bx^*\rangle+\mu^2\|Bu_n-Bu^*\|^2\big]\\
&\le \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\big[\|u_n-x^*\|^2\\
&\quad+\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\big]\\
&= \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\|u_n-x^*\|^2\\
&\quad+\gamma_n\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\
&\quad+\gamma_n\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|+\|x_n-x^*\|^2+\gamma_n\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
-\gamma_n\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2 &\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&=\alpha_n\|f(x^*)-x^*\|^2+\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\\
&\quad\times\left(\|x_n-x^*\|-\|x_{n+1}-x^*\|\right)\\
&\le \alpha_n\|f(x^*)-x^*\|^2+\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\\
&\quad\times\|x_n-x_{n+1}\| \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
-\gamma_n\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2 &\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&=\alpha_n\|f(x^*)-x^*\|^2+\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\\
&\quad\times\left(\|x_n-x^*\|-\|x_{n+1}-x^*\|\right)\\
&\le \alpha_n\|f(x^*)-x^*\|^2+\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\\
&\quad\times\|x_n-x_{n+1}\|. \qquad (30)
\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, it follows from Eqs. (29) and (30) that $\|Ay_n-Ay^*\|\to 0$ and $\|Bu_n-Bx^*\|\to 0$ as $n\to\infty$. From Eq. (10), we have

$$\begin{aligned}
\|y_n-y^*\|^2 &= \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|^2\\
&\le \langle (u_n-\mu Bu_n)-(x^*-\mu Bx^*), y_n-y^*\rangle\\
&=\frac{1}{2}\left\{\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2+\|y_n-y^*\|^2\right.\\
&\qquad\left.-\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)-(y_n-y^*)\|^2\right\}\\
&\le\frac{1}{2}\left\{\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-y_n)-\mu(Bu_n-Bx^*)-(x^*-y^*)\|^2\right\}\\
&=\frac{1}{2}\left\{\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-y_n)-(x^*-y^*)\|^2\right.\\
&\qquad\left.+2\mu\left\langle(u_n-y_n)-(x^*-y^*), Bu_n-Bx^*\right\rangle-\mu^2\|Bu_n-Bx^*\|^2\right\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|y_n-y^*\|^2 &\le \|u_n-x^*\|^2-\|(u_n-y_n)-(x^*-y^*)\|^2\\
&\quad+2\mu\left\langle(u_n-y_n)-(x^*-y^*), Bu_n-Bx^*\right\rangle-\mu^2\|Bu_n-Bx^*\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x^*)-x^*\|^2+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|^2+(\alpha_n\alpha+\beta_n)\|x_n-x^*\|^2+\gamma_n\|y_n-y^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|^2+(\alpha_n\alpha+\beta_n+\gamma_n)\|x_n-x^*\|^2\\
&\quad-\gamma_n\|(u_n-y_n)-(x^*-y^*)\|^2+2\gamma_n\mu\left\langle(u_n-y_n)\right.\\
&\quad\left.-(x^*-y^*), Bu_n-Bx^*\right\rangle-\gamma_n\mu^2\|Bu_n-Bx^*\|^2\\
&\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|(u_n-y_n)-(x^*-y^*)\|^2\\
&\quad+2\gamma_n\mu\|(u_n-y_n)-(x^*-y^*)\|\|Bu_n-Bx^*\|,
\end{aligned}$$

which implies that

$$\begin{aligned}\gamma_n\|(u_n-y_n)-(x^*-y^*)\|^2 &\le \alpha_n\|f(x^*)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\&\quad+2\gamma_n\mu\|(u_n-y_n)-(x^*-y^*)\|\|Bu_n-Bx^*\|\\&\le \alpha_n\|f(x^*)-x^*\|^2+2\gamma_n\mu\|(u_n-y_n)\\&\quad-(x^*-y^*)\|\|Bu_n-Bx^*\|+\|x_n-x_{n+1}\|\\&\quad\times\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\|.\end{aligned}\tag{31}$$

From Eq. (31), $\alpha_n\to 0$, $\|x_n-x_{n+1}\|\to 0$ and $\|Bu_n-Bx^*\|\to 0$ as $n\to\infty$, we have $\|(u_n-y_n)-(x^*-y^*)\|\to 0$ as $n\to\infty$. Moreover, we note that

$$\begin{aligned}\|(y_n-v_n)+(x^*-y^*)\|^2 &= \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-[P_C(y_n-\lambda Ay_n)\\&\quad-P_C(y^*-\lambda Ay^*)]+\lambda(Ay_n-Ay^*)\|^2\\&\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-[P_C(y_n-\lambda Ay_n)\\&\quad-P_C(y^*-\lambda Ay^*)]\|^2+2\lambda\langle Ay_n-Ay^*,(y_n-v_n)+(x^*-y^*)\rangle\\&\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2-\|P_C(y_n-\lambda Ay_n)\\&\quad-P_C(y^*-\lambda Ay^*)\|^2+2\lambda\|Ay_n-Ay^*\|\|(y_n-v_n)+(x^*-y^*)\|\\&\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2\\&\quad-\|SP_C(y_n-\lambda Ay_n)-SP_C(y^*-\lambda Ay^*)\|^2\\&\quad+2\lambda\|Ay_n-Ay^*\|\|(y_n-v_n)+(x^*-y^*)\|\\&= \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2-\|Sv_n-Sx^*\|^2\\&\quad+2\lambda\|Ay_n-Ay^*\|\|(y_n-v_n)+(x^*-y^*)\|\\&\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-(Sv_n-x^*)\|\\&\quad\times(\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|+\|Sv_n-x^*\|)\\&\quad+2\lambda\|Ay_n-Ay^*\|\|(y_n-v_n)+(x^*-y^*)\|\\&= \|u_n-Sv_n+x^*-y^*-(u_n-y_n)-\lambda(Ay_n-Ay^*)\|\\&\quad\times(\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|+\|Sv_n-x^*\|)\\&\quad+2\lambda\|Ay_n-Ay^*\|\|(y_n-v_n)+(x^*-y^*)\|.\end{aligned}\tag{32}$$

Since $\|Sv_n-u_n\|\to 0$, $\|(u_n-y_n)-(x^*-y^*)\|\to 0$ and $\|Ay_n-Ay^*\|\to 0$ as $n\to\infty$, it follows from Eq. (32) that $\|(y_n-v_n)+(x^*-y^*)\|\to 0$ as $n\to\infty$. This implies that

$$\begin{aligned}\|Sv_n-v_n\| &\le \|Sv_n-u_n\|+\|(u_n-y_n)+(x^*-y^*)\|\\&\quad+\|(y_n-v_n)+(x^*-y^*)\|\to 0,\ n\to\infty.\end{aligned}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0,x_n-z_0\rangle\le 0,$$

where $z_0=P_{F(S)\cap\Omega\cap EP(F)}f(z_0)$. To show this inequality, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0,Sv_n-z_0\rangle=\lim_{i\to\infty}\langle f(z_0)-z_0,Sv_{n_i}-z_0\rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i}\rightharpoonup z$. From $\|Sv_n-v_n\|\to 0$, we obtain $Sv_{n_i}\rightharpoonup z$. By the same argument as that in the proof of [12,Theorem 3.1, p. 555], we can show that $z \in EP(F)$. By Lemma 3, we obtain $z \in F(S)$. Finally, by the same argument as that in the proof of [4,Theorem 3.1, p. 385], we can show that $z \in \Omega$. Hence $z \in F(S) \cap \Omega \cap EP(F)$. Now from Eq. (11), we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle f(z_0) - z_0, Sv_n - z_0\rangle \\ &= \lim_{i\to\infty} \langle f(z_0) - z_0, Sv_{n_i} - z_0\rangle \\ &= \langle f(z_0) - z_0, z - z_0\rangle \le 0. \end{aligned} \tag{33}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - z_0\|^2 \\ &\le \|\beta_n(x_n - z_0) + \gamma_n(S_n v_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &\le [\beta_n\|x_n - z_0\|^2 + \gamma_n\|x_n - z_0\|^2] + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\ &\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\alpha\left(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\right) \\ &\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - z_0\|^2 \le \left(1 - \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}\right)\|x_n - z_0\|^2 + \frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n - z_0\|^2 \\ + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle. \end{aligned}$$

Finally by Eq. (33) and Lemma 4, we conclude that $\{x_n\}$ converges to z_0. This completes the proof. □

Corollary 1 (Ceng et al. [4]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B : C \to H$ be the α and β−inverse-strongly monotone mappings, respectively and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{aligned} y_n &= P_C(x_n - \mu B x_n) \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \end{aligned} \tag{34}$$

for all $n \in \mathbf{N}$, where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to $\bar{x} \in F(S) \cap \Omega$ where $\bar{x} = P_{F(S)\cap\Omega} u$ and $(\bar{x}, \bar{y})$ is a solution of problem (5) such that $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Proof Put $S_n \equiv S$ for all $n \in \mathbf{N}$, $f(x) = u := x_1$ for all $x \in H$ and $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$, we get $u_n = x_n$ in Theorem 1. Then, from Theorem 1 the sequence $\{x_n\}$ generated in Corollary 1 converges strongly to $\bar{x} = P_{F(S)\cap\Omega} u$ and $(\bar{x}, \bar{y})$ is a solution of problem Eq. (5) such that $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$. □

Corollary 2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and let $A, B : C \to H$ be the α and $\beta-$inverse-strongly monotone mappings, respectively such that $\Omega \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ y_n = P_C(u_n - \mu B u_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C(y_n - \lambda A y_n), \end{aligned} \tag{35}$$

for all $n \in \mathbf{N}$, where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfying

- (i) $\alpha_n + \beta_n + \gamma_n = 1$,
- (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
- (iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

then $\{x_n\}$ converges strongly to $\bar{x} \in \Omega \cap EP(F)$ where $\bar{x} = P_{\Omega\cap EP(F)} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of problem Eq. (5) such that $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Corollary 3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and let $A : C \to H$ be an $\alpha-$inverse-strongly monotone mappings. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that satisfies condition Eq. (13) and $\cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ y_n = P_C(u_n - \mu A u_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(y_n - \lambda A y_n), \end{aligned} \tag{36}$$

for all $n \in \mathbf{N}$, where $\lambda, \mu \in (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfying

- (i) $\alpha_n + \beta_n + \gamma_n = 1$,
- (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
- (iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

If S is a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ $\forall x \in C$ and $F(S) = \cap_{n=1}^{\infty} F(S_n)$, then $\{x_n\}$ converges strongly to $\bar{x} \in F(S) \cap \Omega \cap EP(F)$ where $\bar{x} = P_{F(S)\cap\Omega\cap EP(F)} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of problem Eq. (6) such that $\bar{y} = P_C(\bar{x} - \mu A\bar{x})$.

4 Applications

Using Theorem 1, we prove three theorems in Hilbert space.

Let $T_1, T_2, \ldots$ be an infinite sequence of mappings of C into itself and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbf{N}$. Then for any $n \in \mathbf{N}$, Takahashi [16] (see also

[14], [17]) defined a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ &\cdot \\ &\cdot \\ &\cdot \\ U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ &\cdot \\ &\cdot \\ &\cdot \\ U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{aligned}$$

Such a mapping W_n is called the *W-mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and λ_n, $\lambda_{n-1}, \ldots, \lambda_1$.

In the following Lemma, we can see the prove in Shimoji and Takahashi [14] and Chang et al.[5].

Lemma 10 ([14] and [5]) *Let C be a nonempty closed convex subset of a Banach space E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\lambda_i\}_{i=1}^{\infty}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$, $\forall i \geq 1$. Then:*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{\infty} F(T_i)$ for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *the mapping $U : C \to C$ defined by*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \;\; x \in C$$

is nonexpansive mapping satisfying $F(U) = \cap_{i=1}^{\infty} F(T_i)$and it is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$;
(4) $\lim_{m,n\to\infty} \sup_{x\in K} \|W_m x - W_n x\| = 0$ *for any bounded subset K of E.*

Setting $S_n \equiv W_n$ in Theorem 1 and using Lemma 10 we obtain the next theorem.

Theorem 2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and let $A, B : C \to H$ be the α and $\beta-$inverse-strongly monotone mappings, respectively. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \cap EP(F) \neq \emptyset$. Let a and b be real numbers with $0 < a \leq b < 1$ and let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$ for every $n = 1, 2, \ldots$. Let W_n be a W-mappings of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$, $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Let U defined by $Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ for every $x \in C$.*

Suppose $x_1 = u \in C$ *and* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are given by*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;$$
$$y_n = P_C(u_n - \mu B u_n)$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C(y_n - \lambda A y_n), \quad (37)$$

for all $n \in \mathbf{N}$, *where* $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are three sequences in* $[0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfying*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $\bar{x} \in F(U) \cap \Omega \cap EP(F)$ *where* $\bar{x} = P_{F(U)\cap\Omega\cap EP(F)} f(\bar{x})$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (5) *such that* $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

A mapping $T : C \to C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \quad \text{for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay\rangle + \|Ax - Ay\|^2.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1 - k}{2}\|Ax - Ay\|^2.$$

Then, A is $\frac{1-k}{2}-$ inverse strongly monotone.

Now, using Theorem 1, we state a strong convergence theorem for a pair of a countable family of nonexpansive mappings and strictly pseudocontractive mapping.

Theorem 3 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C \to \mathbf{R}$ *satisfying (A1)-(A4) and let* $T, V : C \to C$ *are strictly pseudocontractive with constants* k, l, *respectively. Let* f *be a contraction of* H *into itself with* $\alpha \in (0, 1)$ *and let* $\{S_n\}$ *be a sequence of nonexpansive mappings of* C *into itself such that satisfies condition Eq.* (13) *and* $\cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap EP(F) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are given by*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;$$
$$y_n = (1 - \mu)u_n + \mu V u_n$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n((1 - \lambda)y_n + \lambda T y_n), \quad (38)$$

for all $n \in \mathbf{N}$, *where* $\lambda \in (0, 1 - k)$, $\mu \in (0, 1 - l)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are three sequences in* $[0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfying*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

If S is a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x \forall x \in C$ and $F(S) = \cap_{n=1}^{\infty} F(S_n)$, then $\{x_n\}$ converges strongly to $\bar{x} \in F(S) \cap \Omega \cap EP(F)$ where $\bar{x} = P_{F(S)\cap\Omega\cap EP(F)} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of problem Eq. (5) such that $\bar{y} = (1-\mu)\bar{x} + \mu V\bar{x}$.

Proof Put $A = I - T$ and $B = I - V$. Then A is $\frac{1-k}{2}-$inverse-strongly monotone and B is $\frac{1-l}{2}-$inverse-strongly monotone, respectively. We have that

$$P_C(u_n - \lambda_n A u_n) = (1-\mu)u_n + \mu V u_n$$

and

$$P_C(y_n - \lambda A y_n) = (1-\lambda)y_n + \lambda T y_n.$$

Therefore, by Theorem 1 and Eq. (38), the conclusion follows. □

Theorem 4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and let $A : H \to H$ be an $\alpha-$inverse-strongly monotone mapping. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and $l\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that satisfies condition Eq. (13) and $\cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C;$$
$$y_n = u_n + \lambda A u_n$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n(y_n - \lambda A y_n), \tag{39}$$

for all $n \in \mathbf{N}$, where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfying

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

If S is a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x \forall x \in C$ and $F(S) = \cap_{n=1}^{\infty} F(S_n)$, then $\{x_n\}$ converges strongly to $\bar{x} \in F(S) \cap A^{-1}0 \cap EP(F)$ where $\bar{x} = P_{F(S)\cap A^{-1}0\cap EP(F)} f(\bar{x})$.

Proof Put $\lambda = \mu$, $C = H$, $B = A$ and $P_H = I$. By the same argument as that in the proof of [4,Theorem 4.1, p. 388], we can show that $A^{-1}0 = \Omega$ and

$$\text{problem}(5) \Leftrightarrow \text{problem}(6) \Leftrightarrow VI(A, H).$$

Thus, by Theorem 1 we obtain the desired result. □

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